COMMUNICATIONS IN ALGEBRA, 28(5), 2359–2366 (2000)

ON CHAINED OVERRINGS OF PSEUDO-VALUATION RINGS

Ayman Badawi
Department of Mathematics
Birzeit University, Box 14
Birzeit, WestBank, Palestine, via Israel

ABSTRACT. A prime ideal P of a commutative ring R with identity is called strongly prime if aP and bR are comparable for every a, b in R. If every prime ideal of R is strongly prime, then R is called a pseudo-valuation ring. It is well-known that a (valuation) chained overring of a Prufer domain R is of the form R_P for some prime ideal P of R. In this paper, we show that this statement is valid for a certain class of chained overrings of a pseudo-valuation ring.

1. INTRODUCTION

Throughout this paper, all rings are commutative with identity and if R is a ring, then Z(R) denotes the set of zerodivisors of R and T denotes the total quotient ring of R. We say a ring A is an overring of a ring R if A is between R and T. Recall that a ring R is called a chained ring if the principal ideals of R are linearly ordered, that is, if for every $a, b \in R$ either $a|b$ or $b|a$. It is well-known that a chained overring of a Prufer domain R is of the form R_P (see [9, Theorem 65]) for some prime ideal P of R. In this paper, we show that this statement is still valid for a certain class of chained overrings of a pseudo-valuation ring. Recall from [5] that a prime ideal P of a ring R is called a strongly prime ideal if aP and bR are comparable for all $a, b \in R$. If R is an integral domain, this is equivalent to the original definition of strongly prime introduced by Hedstrom and Houston in [8]. If every prime ideal of a ring R is strongly prime, we say that R is a pseudo-valuation ring, abbreviated a PVR. It is easy to see that a PVR is quasilocal, see [5, Lemma 1].

2. RESULTS

We start with the following lemma.

Lemma 1. Let R be a PVR and let $a, b \in R$. If $a \in Z(R)$ and b is a nonzerodivisor of R, then $b|a$. In particular, if $c/d \in T \setminus R$ for some $c, d \in R$, then c is a nonzerodivisor of R and therefore $d/c \in T$.

Proof. Deny. Let M be the maximal ideal of R. Since M is strongly prime and b does not divide a, we must have $bM \subset aR$. Hence, $b^2 = ac$ for some c in R, which is impossible since b^2 is a nonzerodivisor of R and $a \in Z(R)$. Thus, our denial is invalid and $b|a$. ■

The following lemma is trivial, but it is needed in the proof of our main result.

Lemma 2. Let R be a PVR and let A be an overring of R. Then $Z(R) = Z(A)$.

Proof. This is clear by Lemma 1. ■

Theorem 3. Let R be a PVR with maximal ideal M, and let V be a chained overring of R with the maximal ideal N. If $P = N \cap R$ is different from M, then $V = R_P$.

Proof. By Lemma 2, $Z(R) \subset P$. Hence, if $s \in R \setminus P$, then s is a nonzerodivisor of R and $s^{-1} = 1/s \in T$. Now, for any $s \in R \setminus P$, we must have $s^{-1} \in V$, for otherwise $s \in N$ and so $s \in P$. Thus, $R_P \subset V$. Now, we show that $V \subset R_P$. Since P is a nonmaximal prime ideal of R, we note that R_P is a chained ring by [5, Theorem 12]. Suppose that there is a $v \in V$ and v is not in R_P. Write $v = a/s$ for some $a, s \in R$. Since v is not in R_P, $v \in T \setminus R$. Hence, a is a nonzerodivisor of R by Lemma 1 and $v^{-1} \in T$. Since R_P is a chained ring and v is not in R_P, we must have $v^{-1} = s/a \in R_P$. Thus, we may assume $a \notin P$. Since $v^{-1} \in R_P$ and v is not in R_P, we must have $s \in P$, for otherwise, $v^{-1} = s/a$ would be a unit in R_P and $v \in R_P$, which we assumed is not the case. Since $s \in P$, we must have $s \in N$ and $sv \in N$. But $a = sv \in P$, a contradiction. Thus, $V \subset R_P$. Hence, $V = R_P$. ■

It was shown in [5, Lemma 20] that if R is a PVR with maximal ideal M and B is an overring of R containing an element of the form $1/s$ for some nonzerodivisor s of M, then B is a chained ring. In view of Theorem 3, now we can show that such an overring of R is of the form R_P for some prime ideal P of R.

Corollary 4. Let R be a PVR with maximal ideal M, and B be an overring of R containing an element of the form $1/s$ for some nonzerodivisor s of M. Then B is a chained ring of the form R_P for some prime ideal P of R.

Proof. By [5, Lemma 20] B is a chained ring. Let N be the maximal ideal of B. Since B contains an element of the form $1/s$ for some nonzerodivisor s of M, s is not in N. Hence, $N \cap R$ is different from the maximal ideal of R. Thus, $B = R_P$ where $P = N \cap R$ by Theorem 3. ■

It was shown in [2, Proposition 4.3] that if P is a nonmaximal strongly prime ideal of an integral domain R, then $P : P$ is valuation domain. Since P is divided (comparable to every principal ideal of R) by [5, Lemma 1(a)] and nonmaximal, $P : P = \{ x \in T : xP \subset P \}$ contains an element of the form $1/s$ for some nonunit $s \in M \setminus P$. Hence, by Corollary 4, $P : P = R_P$. Thus, we have :

Corollary 5. Let P be a nonmaximal strongly prime ideal of an integral domain R. Then $P : P = R_P$ is a valuation domain. ■

Recall that an ideal of R is called regular if it contains a nonzerodivisor of R. If every regular ideal of R is generated by its set of nonzerodivisors, then R is called a Marot ring. We have the following result.

Proposition 6. Let R be a PVR. Then :

(1) R is a Marot ring.

(2) Z(R) is a prime ideal of R and $T = R_{Z(R)}$.

(3) If R ≠ T, then T is a chained ring.

Proof. (1). This is clear by Lemma 1.**(2).** Since the prime ideals of R are linearly ordered by [5, Lemma 1(a)] and Z(R) is a union of prime ideals of R, Z(R) is a prime ideal of R and hence $T = R_{Z(R)}$.**(3).** If R ≠ T, then Z(R) is a nonmaximal ideal of R. Hence, $T = R_{Z(R)}$ is a chained ring by [5, Theorem 12]. ■

We say an overring B of R is a valuation overring of R if there is an ideal J of B such that for each $t \in T \setminus B$ there is an element $r \in J$ such that $rt \in B \setminus J$. See [9] for more information.

Proposition 7. Let R be a PVR which is not its own total quotient ring, and let B be an overring of R. Then the following are equivalent :

(1) B is a chained overring of R.

(2) B is a valuation overring of R.

Proof. There is nothing to prove if R = T, so we may assume thatR ≠ T. **(1)⟹(2).** This is clear by [9, Theorem 5.1]. **(2)⟹(1).** Since T is a chained ring by Proposition 6(3) and $Z(R) = Z(T) \subset B$ by Lemma 2, B is a chained overring of R by [9, Theorem 23.2]. ■

Now, we state the main result in this paper.

Theorem 8. Let R be a PVR with maximal ideal M. Then the following are equivalent:

(1) Every overring of R is a PVR.

(2) Every chained overring of R other than M : M is of the form R_P for some nonmaximal prime ideal P of R.

(3) M : M is the integral closure of R in T.

Proof. There is nothing to prove if R = T, so we may assume R ≠ T. Since $M : M = \{ x \in T : xM \subset M \}$ is a chained ring with maximal ideal M by [5, Theorem 8], it is the only valuation overring of R that has maximal ideal M (see [9, Theorem 5.1]). Hence M : M is the only chained overring of R that has maximal ideal M by Proposition 7. **(1) ⟺ (3)**. This is clear by [5, Theorem 21]. **(1)⟹(2)**. Since every subring of M : M containing R is a PVR with maximal ideal M by [7, Corollary 18] and M : M is the only chained overring of R that can have M as a maximal ideal, each chained overring of R other than M : M contains an element of the form 1/s where s is a nonzerodivisor of M and thus each is of the form R_P for some prime ideal P of R by Corollary 4. **(2) ⟹ (3)**. First, R is a Marot ring by Proposition 6. Thus, by [8, Theorem 9.3], the integral closure of R in T is the intersection of the valuation overrings of R. By Proposition 7, each valuation overring of R is chained, so except possibly for M : M, each is of the form R_P for some prime ideal P of R. All such rings contain M:M. Therefore , the integral closure of R in T is M : M. ∎

An immediate consequence of the above theorem is the following corollary.

Corollary 9. Let R be a PVR with maximal ideal M and integral closure R' such that $R' \neq M : M$. Then there exists a chained overring W of R such that $R' \subset W \subsetneq M : M$, and W is not of the form R_P for some prime ideal P of R.

Example 10. David F. Anderson provided us with a concrete example of a PVR R that has a valuation overring which is not of the form R_P for some prime ideal P of R. Let $\mathbb{R}$ be the set of real numbers and $\mathbb{C}$ be the set of complex numbers. Set $V = \mathbb{C}(t) + X\mathbb{C}(t)[[X]]$ is a valuation (chained) domain with maximal ideal $M = X\mathbb{C}(t)[[X]]$, and $R = \mathbb{R} + X\mathbb{C}(t)[[X]]$ is a PVR with maximal ideal M. Then $W = \mathbb{C}[t]_{(t)} + X\mathbb{C}(t)[[x]]$ is a valuation (chained) overring of R which is not of the form of R_P for some prime ideal P of R. Observe that $R' = \mathbb{C} + X\mathbb{C}(t)[[X]] \subset W \subsetneq M : M = V$.

ACKNOWLEDGEMENT

My sincere gratitude goes to the referee for his many corrections, in particular, for his remark that (2) implies (3) in Theorem 8. I would like to thank the Department of Mathematics at Vrije University for their hospitality, especially Professor E. J. Ditters for his many helpful communications. Also, I am very grateful to Professor David F. Anderson for providing us with Example 10.

REFERENCES

[1] D.F. Anderson, Comparability of ideals and valuation overrings, Houston J. Math. 5(1979), 451-463.

[2] D.F. Anderson, When the dual of an ideal is a ring, Houston J. Math. 9(1983), 325-332.

[3] A. Badawi, A visit to valuation and pseudo-valuation domains, Zero-Dimensional Commutative Rings, Lecture Notes Pure Appl. Math., Vol. 171(1995), 155-161. Marcel Dekker, New York/Basel.

[4] A. Badawi, On domains which have prime ideals that are linearly ordered, Comm. Algebra 23 (1995), 4365-4373.

[5] A. Badawi, D. F. Anderson, D. E. Dobbs, Pseudo-valuation rings, Commutative Ring Theory, Lecture Notes Pure Appl. Math., Vol. 185 (1997), 57-67. Marcel Dekker, New York/Basel.

[6] A. Badawi, On comparability of ideals of commutative rings, Comm. Algebra 26 (1998), 4365-4373.

[7] A. Badawi, Remarks on pseudo-valuation rings, Comm. Algebra 28 (2000), 2343-2358.

[8] J. R. Hedstrom, E. G. Houston, Pseudo-valuation domains, Pac. J. Math. 75(1978), 137-147.

[9] J. Huckaba, Commutative Rings with Zero Divisors, Marcel Dekker, New York/Basel 1988.

[10] I. Kaplansky, Commutative Rings, rev. ed., Univ. Chicago Press, Chicago, 1974.

Received: March 1998

Revised: May 1999